期中模拟卷解析

1.$(3+3)$用$\varepsilon - N$语言叙述数列a_n不收敛到a的定义，并按定义证明

$$\lim_{n\to\infty}\frac{n^2+1}{2n^2-7n}=\frac{1}{2}$$

a_n不收敛到$a \iff \exists\varepsilon_0>0,$对于$\forall N>0,\exists n_0>N,$有$|a_{n_0}-a|\geq\varepsilon_0$

$\forall\varepsilon>0,\exists N=\max\{2,\frac{2}{\varepsilon}+\frac{7}{2}\},\forall n>N,$有：

$$|\frac{n^2+1}{2n^2-7n}-\frac{1}{2}|=|\frac{7n+2}{2(2n^2-7n)}|<\frac{1}{2}\frac{8n}{2n^2-7n}=\frac{4}{2n-7}<\varepsilon$$

$\therefore\lim\limits_{n\to\infty}\frac{n^2+1}{2n^2-7n}=\frac{1}{2}$

2.(6)计算$\lim\limits_{x\to0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

1^∞化成指数

$\exp\{\frac{1}{x}\ln\frac{a^x+b^x+c^x}{3}\}$

$\sim\exp\{\frac{1}{x}(\frac{a^x+b^x+c^x}{3}-1)\}$

$=\exp\{\frac{a^x-1+b^x-1+c^x-1}{3x}\}$

$\sim\exp\{\frac{1}{3}(\ln a+\ln b+\ln c)\}$

$=\sqrt[3]{abc}$

3.(8)计算$\lim\limits_{x\to0}\frac{(3+2\sin x)^x-3^x}{\tan^2 x}$

$\frac{3^x((1+\frac{2\sin x}{3})^x-1)}{\tan^2 x}\sim3^x\frac{x\ln(1+\frac{2\sin x}{3})}{\tan^2 x}$

$\sim3^x\frac{2x\sin x}{3\tan^2 x}\to\frac{2}{3}(x\to0)$

4.(6) $\lim\limits_{x\to+\infty}\left(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right)$

$=\lim\limits_{x\to+\infty}\frac{2x}{\sqrt{1+x+x^2}+\sqrt{1-x+x^2}}\to1(x\to+\infty)$

严谨点应该把分子除下去

5.(8)求$\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}(\frac{1}{\sqrt[k]{n^k+1}}+\frac{1}{\sqrt[k]{n^k-1}})$

$n\leq\sqrt[k]{n^k+1}\leq n+1$

$n-1 \le \sqrt[k]{n^k-1} \le n$

answer: 2

6.(8)求$f(x)=\dfrac{\left(e^{\frac{1}{x}}+e\right)\tan x}{x\left(e^{\frac{1}{x}}-e\right)}$的间断点, 并判断类型

分母等于0：$x=0,1;\tan x$无意义：$x=\dfrac{\pi}{2}+k\pi(k\in\mathbb{Z})$

求极限$x=0$跳跃，其余无穷间断点

7.(8)$f(x)=\begin{cases}\dfrac{x}{1+e^{\frac{1}{x}}}, x\neq 0\\ 0, x=0\end{cases}$，求$f(x)$在零点的左右导数，判断在零点是否可导

$f'_+(0)=\lim\limits_{x\to0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0^+}\dfrac{1}{1+e^{\frac{1}{x}}}=0$

$f'_-(0)=\lim\limits_{x\to0^-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0^-}\dfrac{1}{1+e^{\frac{1}{x}}}=1$

8.(6)$f(x)=(e^x+\log_3 x)\arctan\dfrac{1+x}{1-x}$，求$f(x)$的导函数

$f'(x)=(e^x+\dfrac{1}{x\ln 3})\arctan\dfrac{1+x}{1-x}+(e^x+\log_3 x)\dfrac{1}{1+(\frac{1+x}{1-x})^2}\dfrac{2}{(x-1)^2}$

9.(4+4)已知x和y满足参数方程$\begin{cases}x=a(\cos t+t\sin t)\\ y=a(\sin t-t\cos t)\end{cases}$

(1)求此曲线$y=y(x)$的在任意一点的法线到原点的距离, (2)求$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{t\sin t}{t\cos t}=\tan t$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{1+\tan^2 t}{at\cos t}$

法线方程：$y-a(\sin t-t\cos t)=-\dfrac{\cos t}{\sin t}(x-a(\cos t+t\sin t))$

$\sin ty+\cos tx-a=0$

$d=|a|$

10.(4+4)已知$2y\sin x+x\ln y=0$确定了$y=f(x)$的函数，求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$与$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$

对x求导：$2\sin x\cdot y'+2y\cos x+\ln y+\dfrac{x}{y}y'=0$

再对x求导：$2\cos xy'+2\sin xy''+2y'\cos x-2y\sin x+\dfrac{y'}{y}+\dfrac{y-xy'}{y^2}y'+\dfrac{x}{y}y''=0$

$y' = -\dfrac{2y\cos x + \ln y}{2\sin x + \dfrac{x}{y}}$

$y'' = \ldots$

11.(8)求$y = \dfrac{x^n}{1-x}$的n阶导数

$y = -\dfrac{x^n-1}{x-1} - \dfrac{1}{x-1} = -(1+x+\ldots+x^{n-1}) - \dfrac{1}{x-1}$

$y^{(n)} = (-1)^{n+1}n!(x-1)^{-n-1}$

12.$(6+2)$已知$c>0, a_1 = \dfrac{c}{2}, a_{n+1} = \dfrac{c}{2} + \dfrac{a_n^2}{2}$, 讨论$a_n$的收敛情况, 若收敛, 求出其极限

$a_{n+1} - a_n = \dfrac{1}{2}(a_n^2 - a_{n-1}^2) = \dfrac{1}{2}(a_n - a_{n-1})(a_n + a_{n-1})$

所以$a_{n+1} - a_n$和$a_n - a_{n-1}$同号

$a_2 - a_1 = \dfrac{c^2}{8} > 0 \Rightarrow a_n$递增

$c > 1$时

假设有上界，则极限存在

$A^2 - 2A + c = 0$无实数解，矛盾，所以无上界，$a_n \to +\infty (n \to +\infty)$

$c \le 1$时

归纳证明$a_n < 1 \Leftarrow a_{n+1} < \dfrac{1}{2} + \dfrac{1}{2} = 1$

由单调有界定理：a_n收敛

设$\lim\limits_{n\to\infty} a_n = A$

$A = \dfrac{c}{2} + \dfrac{A^2}{2} \Rightarrow A = 1 - \sqrt{1-c}$

13.(6)设$\lim\limits_{n\to\infty} \dfrac{a_1 + a_2 + \ldots + a_n}{n} = a$, 证明：

$$\lim_{n\to\infty} \frac{a_n}{n} = 0$$

注意此题不能用$Stolz$

$\lim\limits_{n\to\infty} \dfrac{a_n}{n} = \lim\limits_{n\to\infty} \dfrac{a_1 + a_2 + \ldots + a_n}{n} - \dfrac{a_1 + a_2 + \ldots + a_{n-1}}{n-1} \cdot \dfrac{n-1}{n} = a - a = 0$

14.$(3+3)$已知$f(x)$在$[0,1]$上连续，在$(0,1)$上可导，$f(0) = f(1) = 0$，$f(\dfrac{1}{2}) = 1$.证明：

(1)$\exists \varepsilon \in (\dfrac{1}{2}, 1)$, 使得$f(\varepsilon) = \varepsilon$

$g(x) = f(x) - x$在$(\dfrac{1}{2}, 1)$上存在零点

$(2)\forall\gamma\in\mathbb{R},\exists\rho\in(0,\varepsilon)$,使得$f'(\rho)-\gamma(f(\rho)-\rho)=1$

$h(x)=e^{-\gamma x}(f(x)-x)$

$h'(x)=e^{-\gamma x}(f'(x)-1-\gamma(f(x)-x))$

高中导数题的构造函数，本质上是微分方程